TRANSIENT HEAT CONDUCTION IN HOLLOW SPHERES WITH A MOVING INNER BOUNDARY

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The finite integral transform method is used to obtain the solution of unsteady heat conduction problems for a hollow sphere with a moving internal boundary and various boundary conditions at the outer surface. For the solution of the problems of interest integral transform formulas are presented with kernels (16), (20), and (24) and the corresponding inversion formulas (18), (22), (26), (29) and characteristic equations (17), (21), (25), (28), (31), (33).

Using a method analogous to that used in [1, 2], we shall obtain solutions to spherically-symmetric problems involving a moving inner boundary and various outer boundary conditions.

Up to the moment when the boundary begins to move, the mathematical formulation of the problem is

$$\frac{\partial t}{\partial \tau} = a\left(\frac{\partial^2 t}{\partial r^2} + \frac{2}{r}\frac{\partial t}{\partial r}\right), \; R_1 < r < R_2, \; \tau > 0, \tag{1}$$

$$t(r, \tau)|_{\tau=0} = 0, \tag{2}$$

$$\left.\frac{\partial t(r, \tau)}{\partial r}\right|_{r=R_1} = -\frac{q_1(\tau)}{\lambda}, \tag{3}$$

$$\alpha(\tau)\left.\frac{\partial t}{\partial r}\right|_{r=R_2} + \beta(\tau)t(r, \tau)|_{r=R_2} + \gamma(\tau) = 0. \tag{4}$$

Using the appropriate integral transform, one can obtain the solution to (1)-(4). Assume this solution to be $\Theta = r^{1/2} f_0(r, \tau)$, where

$$\Theta = r^{1/2} t(r, \tau). \tag{5}$$

From the condition

$$t_{(\varphi)} = f_0(r, \tau_0) \tag{6}$$

one can find $\tau_0 = \Phi(R_1, t_{(\varphi)})$, i.e. the time at which the boundary begins to move. From the time $\tau = \tau_0$ on, the boundary $r = R_1$ moves according to the law $r = s(\tau)$. In that case the mathematical formulation will differ from (1)-(4), first because of the conditions

$$t(r, \tau)|_{\tau=\tau_0} = f_0(r, \tau_0), \tag{7}$$

$$t(r, \tau)|_{r=s(\tau)} = t_{(\varphi)} \tag{8}$$

and, second, because of the additional heat-balance condition at the moving boundary

$$\lambda \left.\frac{\partial t(r, \tau)}{\partial r}\right|_{r=s(\tau)} = -\frac{Q_0(\tau)}{4\pi r^2} + \rho F \frac{ds}{d\tau}. \tag{9}$$

Now it is required to find the temperature field for $\tau > \tau_0$, $s(\tau) \le r \le R_2$ and the law of motion of the boundary. We shall divide the arbitrary time interval $T = T_0 - \tau_0$ into n parts, as in [1, 2], corresponding to the times $\tau_1, \tau_2, \ldots, \tau_0, \tau_n = T_0$, $\Delta\tau_i = \tau_{i+1} - \tau_i$. The time τ_i corresponds to the point O_i $[r_i = s(\tau_i)]$ on the r axis, and we shall assume that the point O_i is stationary for $\tau_i < \tau < \tau_{i+1}$ and jumps instantaneously to O_{i+1} at $\tau = \tau_{i+1}$. As a result we obtain a step-like $s_n(\tau)$ instead of $s(\tau)$. It can be proved easily that the function $t_i(r, \tau)$ satisfies (1) with $r > r_i$, $\tau > \tau_i$, the initial condition $t_i(r, \tau) = f_i(r, \tau_i) = t_{i-1}(r, \tau_i)$, boundary condition (8) at $r = s_i = s(\tau_i)$, and boundary condition (4).

Using (5), one can solve the above problem by the method of finite integral transforms. Assuming the solution to be

$$t_i(r, \tau) = f_i(r, \tau_i) \tag{10}$$

and taking into account that

$$f_i(r, \tau_i) = t_{i-1}(r, \tau_i), \tag{11}$$

we can express t_{i-1} in (10) in terms of t_{i-2}, then express t_{i-2} in terms of t_{i-3}, etc. The result is

$$t_i(r, \tau) = F_i[f_0(r, \tau_0), r, R_2, r_i, r_{i-1}, \ldots, R_1, \tau, \tau_i, \ldots, \tau_0]. \tag{12}$$

Using (12), we can determine the unknown values $r_i = s(\tau_i)$, i. e., the approximate law of motion of the boundary. Rewrite Eq. (9) in the form

$$\frac{ds}{d\tau} = \frac{\lambda}{\rho F}\left.\frac{\partial t_i(r, \tau)}{\partial r}\right|_{r=s_i} + \frac{Q_0(\tau)}{4\pi\rho F s_i^2}. \tag{13}$$

The right-hand side of (13) is a known function of τ. Integrating (13) over τ from τ_i to τ_{i+1} and adding the equations for $i = 0, 1, 2, \ldots, l$ $(l \le n)$, we obtain

$$s(\tau_l) - s(\tau_i) = \sum_{i=0}^{l-1}[s(\tau_{i+1}) - s(\tau_i)] = \sum_{i=0}^{l-1}\int_{\tau_i}^{\tau_{i+1}}\left[\left.\frac{\lambda\,\partial t_i}{\rho F\partial r}\right|_{r=s_i} + \frac{Q_0(\tau)}{4\pi\rho F s_i^2}\right]d\tau. \tag{14}$$

Knowing $s_i(\tau)$ from (12), we obtain the functions $t_i(r, \tau)$. We shall now illustrate this method by means of specific examples.

Hollow sphere with boundary condition of the first kind at the outer surface ($\alpha(\tau) = \beta(\tau) = 1$, $\gamma(\tau) = \varphi_2(\tau)$).

Applying to the function $\Theta(r, \tau)$ the integral transform

$$\overline{\Theta}_{\mu_n}(\tau) = \int_{R_1}^{R_2} r\,\Theta(r, \tau)\,W_0\left(\mu_n\frac{r}{R_1}\right)dr \tag{15}$$

with the kernel

$$W_0\left(\mu_n\frac{r}{R_1}\right) = \frac{2}{\pi\mu_n k^{1/2}}\left(\frac{R_1}{r}\right)^{1/2}\sin\mu_n\left(k - \frac{r}{R_1}\right), \tag{16}$$

where μ_n are the roots of the characteristic equation

$$\operatorname{tg}\mu_n(k-1) = -\mu_n \tag{17}$$

with $k = R_2/R_1$, and using the inversion formula

$$W_0^{-1}[\overline{\Theta}_{\mu_n}(\tau)] = \Theta(r, \tau) = r^{1/2}t(r, \tau) = \frac{\pi^2 k}{R_1^2}\sum_{n=1}^{\infty}\frac{\mu_n^3\overline{\Theta}_{\mu_n}(\tau)}{2\mu_n(k-1) - \sin 2\mu_n(k-1)}W_0\left(\mu_n\frac{r}{R_1}\right), \tag{18}$$

we obtain the solution

$$\Theta(r, \tau) = r^{1/2}t(r, \tau) = \frac{\pi^2 k}{aR_1^{3/2}}\sum_{n=1}^{\infty}\frac{\mu_n^3}{2\mu_n(k-1) - \sin 2\mu_n(k-1)}\times \int_0^{\tau}\left[\frac{R_1}{\lambda}q_1(\theta)W_0(\mu_n) - \mu_n k^{3/2}\varphi_2(\theta)W_0'(\mu_n k)\right]\times \exp[-\mu_n^2(\mathrm{Fo}(1, \tau) - \mathrm{Fo}(1, \theta))]\,d\theta\,W_0\left(\mu_n\frac{r}{R_1}\right). \tag{19}$$

We determine τ_0 from Eq. (6). For the time $\tau \geq \tau_0$ and $r \geq r_i = s(\tau_i) = s_i$ we use transformation (15) with the kernel

$$V_0\left(\delta_{\nu_i}\frac{r}{r_i}\right) = \frac{2}{\pi\delta_{\nu_i}}\left(\frac{r_i}{r}\right)^{1/2}\sin\delta_{\nu_i}\left(\frac{r}{r_i}-1\right), \tag{20}$$

where $\delta_{\nu i}$ are the roots of the characteristic equation

$$\sin\delta_{\nu_i}(k_i - 1) = 0 \tag{21}$$

with $k_i = R_2/R_1$. Using the inversion formula

$$V_0^{-1}[\bar{\Theta}_{\delta_{\nu_i}}(\tau)] = \Theta_i(r,\ \tau) = r^{1/2}t_i(r,\ \tau) = = \frac{\pi^2}{2}\sum_{\nu_i=1}^{\infty}\frac{\delta_{\nu_i}^2\bar{\Theta}_{\delta_{\nu_i}}(\tau)}{r_i^2(k_i-1)}V_0\left(\delta_{\nu_i}\frac{r}{r_i}\right), \tag{22}$$

we obtain the solution

$$\begin{aligned}\Theta_i(r,\ \tau) = \frac{\pi^2}{2}\sum_{\nu_i=1}^{\infty}\frac{\delta_{\nu_i}}{r_i^2(k_i-1)}\{r_i^{5/2}t_{(\varphi)}V_0'(\delta_{\nu_i})\times\\ \times\{1-\exp[-\delta_{\nu_i}^2(\mathrm{Fo}(i,\ \tau)-\mathrm{Fo}(i,\ \tau_i))]\}-\\ -a\delta_{\nu_i}^2k_iR_2^{1/2}V_0'(\delta_{\nu_i}k_i)\times\\ \times\int_{\tau_i}^{\tau}\exp[-\delta_{\nu_i}^2(\mathrm{Fo}(i,\ \tau)-\mathrm{Fo}(i,\ \theta))]\varphi_2(\theta)d\theta+\\ +\delta_{\nu_i}\exp[-\delta_{\nu_i}^2(\mathrm{Fo}(i,\ \tau)-\mathrm{Fo}(i,\ \tau_i))]\times\\ \times\int_{r_i}^{R_2}r^{3/2}f_i(r)V_0\left(\delta_{\nu_i}\frac{r}{r_i}\right)dr\}V_0\left(\delta_{\nu_i}\frac{r}{r_i}\right).\end{aligned} \tag{23}$$

Substituting (23) in (13) we determine $s_i(\tau)$, and substituting $s_i(\tau)$ in (23) we determine $\Theta_i(r,\ \tau)$.

Hollow sphere with boundary condition of the second kind at the outer surface ($\alpha(\tau) = 1$, $\beta(\tau) = 0$, $\gamma(\tau) = \lambda^{-1}q_2(\tau)$, $q_2(\tau) < q_1(\tau)$).

Applying, in this case, integral transform (15) with the kernel

$$\begin{aligned}W_0\left(\mu_n\frac{r}{R_1}\right) = \frac{4}{\pi\mu_n}\left(\frac{R_1}{r}\right)^{1/2}\times\\ \times\left[\sin\mu_n\left(\frac{r}{R_1}-1\right)+\mu_n\cos\mu_n\left(\frac{r}{R_1}-1\right)\right],\end{aligned} \tag{24}$$

where μ_n are the roots of the characteristic equation

$$\mathrm{tg}\,\mu_n(k-1) = (k-1)\mu_n/(1+k\mu_n^2), \tag{25}$$

and using the inversion formula

$$\begin{aligned}W_0^{-1}[\bar{\Theta}_{\mu_n}(\tau)] = \Theta(r,\ \tau) = \\ = \frac{\pi^2}{4R_1^2}\sum_{n=1}^{\infty}[\mu_n^3\bar{\Theta}_{\mu_n}(\tau)W_0(\mu_n r/R_1)]\{2\mu_n[k+\mu_n^2(k-1)]-\\ -(1-\mu_n^2)\sin 2\mu_n(k-1)-2\mu_n\cos 2\mu_n(k-1)\}^{-1},\end{aligned} \tag{26}$$

we obtain the solution

$$\Theta(r,\ \tau) = \pi^3 a/4\lambda R_1^{1/2} \sum_{n=1}^{\infty} \mu_n^3 \int_0^{\tau} [q_1(\theta) W_0(\mu_n) -$$

$$- k^{3/2} q_2(\theta) W_0(\mu_n k)] \exp[-\mu_n^2(\mathrm{Fo}(1,\ \tau) -$$

$$- \mathrm{Fo}(1,\ \theta)] d\theta \{2\mu_n [k + \mu_n^2 (k-1)] - (1-\mu_n^2) \times$$

$$\times \sin 2\mu_n (k-1) - 2\mu_n \cos 2\mu_n (k-1)\}^{-1} W_0(\mu_n r/R_1), \tag{27}$$

while for $\tau \ge \tau_i$ and $r \ge r_i$ we use integral transform (15) with kernel (20) in which δ_{ν_i} are the roots of the equation

$$\mathrm{tg}\, \delta_{\nu_i}(k_i - 1) = k_i \delta_{\nu_i}, \tag{28}$$

and using the inversion formula

$$V_0^{-1}[\bar{\Theta}_{\mu_n}(\tau)] = \Theta(r,\ \tau) =$$

$$= \pi^2 \sum_{\nu_i=1}^{\infty} \frac{\delta_{\nu_i} \bar{\Theta}_{\delta_{\nu_i}}(\tau)}{r_i^2 [2\delta_{\nu_i}(k_i - 1) - \sin 2\delta_{\nu_i}(k_i - 1)]} V_0\left(\delta_{\nu_i} \frac{r}{r_i}\right), \tag{29}$$

we find the solution

$$\Theta_i(r,\ \tau) = \frac{\pi^2}{a\lambda} \sum_{\nu_i=1}^{\infty} \frac{\delta_{\nu_i}^2 V_0(\delta_{\nu_i} r/r_i)}{r_i^2 [2\delta_{\nu_i}(k_i - 1) - \sin 2\delta_{\nu_i}(k_i - 1)]} \times$$

$$\times \{\lambda r_i^{3/2} t_{(\varphi)} V_0'(\delta_{\nu_i}) \{1 - \exp[-\delta_{\nu_i}^2(\mathrm{Fo}(i,\ \tau) - \mathrm{Fo}(i,\ \tau_i))]\} - \tag{30}$$

$$- a\delta_{\nu_i} R_2^{3/2} V_0(\delta_{\nu_i} k_i) \int_{\tau_i}^{\tau} q_2(\theta) \exp[-\delta_{\nu_i}^2 (\mathrm{Fo}(i,\ \tau) - \mathrm{Fo}(i,\ \theta))] d\Theta -$$

$$- a\delta_{\nu_i} \lambda \exp[-\delta_{\nu_i}^2 (\mathrm{Fo}(i,\ \tau) - \mathrm{Fo}(i,\ \tau_i))] \times$$

$$\times \int_{r_i}^{R_2} r^{3/2} f_i(r) V_0\left(\delta_{\nu_i} \frac{r}{r_i}\right) dr\}.$$

The unknown functions are determined in a manner analogous to the preceding case.

Hollow sphere with boundary condition of the third kind at the outer surface ($\alpha(\tau) = 1,\ \gamma(\tau) = 0,\ \beta(\tau) = \alpha/\lambda$).

In this case we use integral transform (15) with kernel (24) in which μ_n are the roots of the equation

$$\mathrm{tg}\, \mu_n (k-1) = \mu_n (k - 1 + \mathrm{Bi}_2)/(1 + k\mu_n^2 - \mathrm{Bi}_2). \tag{31}$$

Using inversion formula (26), we find the solution

$$W_0^{-1}[\bar{\Theta}_{\mu_n}(\tau)] = \Theta(r,\ \tau) = \frac{\pi^3 a}{4\lambda R_1^{1/2}} \times$$

$$\times \sum_{n=1}^{\infty} [\mu_n^3 W_0(\mu_n) W_0(\mu_n r/R_1)] \{2\mu_n [k + \mu_n^2 (k-1)] -$$

$$- (1-\mu_n^2) \sin 2\mu_n (k-1) - 2\mu_n \cos 2\mu_n (k-1)\}^{-1} \times$$

$$\times \int_0^{\tau} q_1(\theta) \exp[-\mu_n^2(\mathrm{Fo}(1,\ \tau) - \mathrm{Fo}(1,\ \theta))] d\theta. \tag{32}$$

For $\tau \ge \tau_0$ we use transformation (15) with kernel (20) in which δ_{ν_i} are the roots of the equation

$$\mathrm{tg}\, \delta_{\nu_i}(k_i - 1) = \delta_{\nu_i} k_i/(1 - \mathrm{Bi}_2). \tag{33}$$

The inversion formula is, in this case, of the form of (29). Hence the solution is

$$V_0^{-1}[\overline{\Theta}_{\delta_{\nu_i}}(\tau)] = \Theta_i(r, \tau) =$$

$$= \pi^2 \sum_{\nu_i=1}^{\infty} \delta_{\nu_i} r_i^{3/2} t_{(\varphi)} V_0'(\delta_{\nu_i}) V_0\left(\delta_{\nu_i} \frac{r}{r_i}\right) \times$$

$$\times r_i^{-2}[2\delta_{\nu_i}(k_i - 1) - \sin 2\delta_{\nu_i}(k_i - 1)]^{-1} \times$$

$$\times \left\{[1 - \exp(-\delta_{\nu_i}^2 \mathrm{Fo}(i, \tau))] + \delta_{\nu_i}^3 \times\right.$$

$$\left.\times \int_{r_i}^{R_2} r^{3/2} f_i(r) V_0\left(\delta_{\nu_i} \frac{r}{r_i}\right) dr\right\}. \tag{34}$$

Analogous methods can be used to solve problems involving phase change at the outer boundary or time-dependent physical properties.

NOMENCLATURE

a, λ – thermal diffusivity and conductivity, respectively; $t_{(\varphi)}$ – temperature of phase transformation; ρ – density; α – heat transfer coefficient; Q – total quantity of heat passing through inner boundary; F – latent heat of phase transformation; $\mathrm{Fo}(1, \tau) = a\tau/R_1^2$, $\mathrm{Fo}(i, \tau) = a\tau/r_i^2$, $\mathrm{Fo}(i, \tau_i) = a\tau_i/r_i^2$ – Fourier numbers; $\mathrm{Bi}_2 = \alpha R_2/\lambda$ – Biot number.

REFERENCES

1. G. A. Surkov and V. I. Krylovich, Inzh.-fiz. zhurn., No. 6, 1964.
2. G. A. Surkov, Inzh.-fiz. zhurn., no. 3, 1964.

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